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On the Distribution of Zeros of Polynomials Orthogonal on the Unit Circle

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Let $\{\Phi_n\}$ be a system of monic polynomials orthogonal on the unit circle with respect to a positive Borel measure μ. It is shown that under fairly mild conditions on the reflection coefficients, $\Phi_n(0)$, the zeros of a subsequence of $\{\Phi_n\}$ are asymptotically uniformly distributed on some circle. The radius of this circle can be found using a Cauchy-Hadamard-type formula. We also characterize the measures μ with the property that the $L^2_{d\mu}$ best polynomial approximants to every function $f \in H^2(\mu)$ converge uniformly on every compact subset of the open unit disc at a geometrically fast rate. © 1990 Academic Press, Inc.

1. INTRODUCTION

Let μ be a positive Borel measure supported on an infinite subset of the complex plane, C. Suppose that

$$\int |t|^n \, d\mu(t) < \infty, \qquad n = 0, 1, \dots. \tag{1.1}$$

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Then the orthonormal polynomials $\phi_n(d\mu)$ are defined uniquely by the conditions

$$\int \phi_n(t)\,\overline{\phi_k(t)}\,d\mu(t)=\delta_{kn} \tag{1.2a}$$

$$\phi_n(t)=\kappa_n t^n+\cdots\in \Pi_n, \qquad \kappa_n>0, \tag{1.2b}$$

where Π_n denotes the class of polynomials of degree at most n. When the support of μ is a finite real interval, then the asymptotic behavior of the zeros of ϕ_n has been thoroughly investigated. Denoting the zeros of ϕ_n by $\{z_{kn}\}_{k=1}^{n}$, we have, for example, the following theorem of Erdős and Freud.

THEOREM 1.1 [4]. *Let the support of μ be the interval $[-1, 1]$. Assume that*

$$\lim_{n\to\infty} \kappa_n^{1/n}=2.$$

Then, for any continuous function f on $[-1, 1]$,

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n} f(z_{kn})=\frac{1}{\pi}\int_{-1}^{1} f(t)(1-t^2)^{-1/2}\,dt. \tag{1.3}$$

The assumptions of this theorem are true, in particular, when the support of μ is the interval $[-1, 1]$ and the Radon–Nykodym derivative of μ is positive almost everywhere. Under the assumptions of this theorem, then, the limiting distribution of the zeros of the orthogonal polynomials is arcsine.

When the support of μ is the unit circle, however, very little is known about the location of the zeros of the orthogonal polynomials [1, 10, 13]. In a recent paper, Nevai and Totik [10] have established certain connections between the recurrence coefficients of these polynomials and their zeros. In this paper, we demonstrate how a lemma in a paper of Blatt, Saff, and Simkani [2] together with some of the estimates in [6] can be used to obtain, in fact, an analogue of Theorem 1.1 in this case. As an application of our theorem, we shall study the behavior of the polynomials of best approximation to functions in the Hardy class H^2 on the unit disc in the metric of the space $L^2_{d\mu}$. When μ is the arclength measure on the unit circle, then these polynomials are simply the partial sums of the Taylor expansion of the function; therefore they converge to the function uniformly on compact subsets of the open unit disc at a geometrically fast rate. Subject to certain mild conditions, we shall characterize those measures μ for which the $L^2_{d\mu}$ best approximation polynomials for every function in H^2

converge geometrically on compact subsets of the open unit disc. Our result concerning the zero distribution can be generalized further, for instance, to the case when the support of μ is a sufficiently smooth closed Jordan curve; but, because of the technical details involved, we defer this generalization to a separate paper along with a general treatment of the zeros of extremal polynomials.

In the next section, we discuss our main results, while the proofs will be given in Section 3.

2. Main Results

First we develop some notation. We assume that μ is a positive Borel measure supported on the unit circle and satisfies condition (1.1). Let Φ_n denote the monic polynomial $\kappa_n^{-1}\phi_n$. If p is a polynomial of degree n, then the polynomial $z^n\overline{p(1/\bar{z})}$ will be denoted by p^*. For $r>0$, let v_r denote the arc-measure $(2\pi)^{-1}\,d\theta$ on the circle $C_r := \{z : |z| = r\}$. When $r=0$, we let v_r be the delta distribution with mass 1 and support $z=0$. If p_n is a polynomial of degree n, then $v(p_n)$ denotes the measure that associates the mass of $1/n$ at each of the n zeros of p_n. A limit of measures will always mean the limit in the weak* topology. Let $\|\cdot\|_E$ denote the sup norm on E. Finally, if μ is a measure, μ' its Radon–Nykodym derivative, and $\log \mu'$ is integrable, then the Szegö function is defined by

$$D(\mu, z) := \exp\left\{\frac{1}{4\pi}\int_0^{2\pi} \log \mu'(t)\,\frac{u+z}{u-z}\,dt\right\}, \qquad u = e^{it}. \tag{2.1}$$

We summarize some of the well-known facts about the orthogonal polynomials on the unit circle in the following

Proposition 2.1 [5, 6, 14]. (a) *The polynomials Φ_n, Φ_n^* satisfy the recurrence relation*

$$\Phi_n^*(z) = \Phi_{n-1}^*(z) + z\overline{\Phi_n(0)}\,\Phi_{n-1}(z), \tag{2.2a}$$

so that

$$\Phi_n^*(z) = 1 + z\sum_{k=0}^{n-1} \overline{\Phi_{k+1}(0)}\,\Phi_k(z). \tag{2.2b}$$

(b) *All the zeros of ϕ_n are in the open unit disc $\{|z|<1\}$, and hence*

$$|\Phi_n(0)| < 1, \qquad n = 1, 2, ... \tag{2.3}$$

(c) *Given any sequence of numbers* a_k *with* $|a_k| < 1$, *there exists a unique measure* μ *such that* $a_k = \Phi_k(0)$, $k = 1, 2, \ldots$.

(d) *We have*

$$|\Phi_n(z)| \leqslant \exp\left(\sum_{k=0}^{n} |\Phi_k(0)|\right), \qquad |z| \leqslant 1, \qquad n = 0, 1, \ldots. \tag{2.4}$$

(e) *The measure* μ *satisfies* $\log \mu' \in L^1[0, 2\pi]$ *if and only if*

$$\lim_{n \to \infty} \phi_n^*(z) = D(\mu, z)^{-1}, \qquad |z| < 1, \tag{2.5}$$

the limit being uniform on compact subsets of the open unit disc.

(f) *If* $\log \mu'$ *is integrable, then* $D(\mu, z)^{-1} \in H^2$.

Nevai and Totik proved, among other things, the following facts.

THEOREM 2.2 [10]. *Let*

$$\limsup_{n \to \infty} |\Phi_n(0)|^{1/n} =: \rho. \tag{2.6}$$

Assume that $\rho < 1$. *Then*

(a) *For each* $\sigma > \rho$, *the number of zeros of* ϕ_n *outside of* C_σ *is bounded independently of* n.

(b) *The function* $D(\mu, z)^{-1}$ *is analytic in the disc* $|z| < 1/\rho$, *and* (2.5) *holds on compact subsets of this disc.*

(c) *The measure* μ *is absolutely continuous and there is a function* g *which is analytic and non-zero on* $|z| < 1/\rho$, *such that* $\mu'(\theta) = |g(e^{i\theta})|^2$, $\theta \in [0, 2\pi]$.

We shall sharpen part (a) of the above theorem by giving the limiting distribution of the zeros of the orthogonal polynomials. Under some mild conditions on μ, we shall also give the distribution of the zeros even for the case $\rho = 1$, where ρ is defined in (2.6), thus completing Theorem 2.2. We note that, for the "Jacobi case," $\mu' = |\sin(\theta/2)|^{2\gamma}$, $\gamma > 0$, we have [3, 9] $\Phi_n(0) = \gamma/(n+\gamma)$, so that $\rho = 1$. The following theorem will imply that, in this case, the zero distributions $\{\nu(\phi_n)\}_{n=1}^{\infty}$ converge to the measure $(2\pi)^{-1}\, d\theta$ on the unit circle.

THEOREM 2.3. *Let* Λ *be any subsequence of positive integers such that*

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} |\Phi_n(0)|^{1/n} = \rho, \tag{2.7}$$

where ρ is defined in (2.6). *If $\rho < 1$, then, in the weak* topology,*

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \nu(\phi_n) = \nu_\rho. \tag{2.8}$$

Suppose that

$$\lim_{n \to \infty} \frac{1}{n} \sum_{k=1}^{n} |\Phi_k(0)| = 0. \tag{2.9}$$

Then (2.8) *holds even in the case when $\rho = 1$.*

Remark 1. If $\rho < 1$, then (2.9) is trivially satisfied. In [11] (cf. [8] also), it is proved that if $\mu' > 0$ almost everywhere on the unit circle, then $\Phi_n(0) \to 0$ as $n \to \infty$, which again implies (2.9). Thus, the condition (2.9) is a fairly weak condition.

Remark 2. After proving Theorem 2.3, we shall demonstrate that when $0 < \Phi_n(0) < 1$, $n = 1, 2, ...$, but (2.9) is not satisfied, then (2.8) does not hold.

Remark 3. In view of the Principle of Descent [7, Theorem 3.8], (2.8) implies that, with the terminology to be defined at the beginning of Section 3,

$$\limsup_{\substack{n \to \infty \\ n \in \Lambda}} |\Phi_n(z)|^{1/n} = \begin{cases} \rho, & \text{for quasi-all } |z| \leqslant \rho \\ |z|, & \text{for quasi-all } |z| > \rho. \end{cases} \tag{2.10}$$

We now turn our attention to the approximation of functions in H^2. If $f \in H^2(\mu)$, we let

$$c_k(\mu, f) := \int f(t)\, \overline{\phi_k(t)}\, d\mu(t), \qquad k = 0, 1, ..., \tag{2.11a}$$

and

$$s_n(\mu, f, z) := \sum_{k=0}^{n-1} c_k(\mu, f)\, \phi_k(z), \qquad z \in \mathbf{C}. \tag{2.11b}$$

A measure μ will be said to have the *geometric convergence property* if for every $f \in H^2(\mu)$ and every compact set $K \subset \{|z| < 1\}$, we have

$$\limsup_{n \to \infty} \|f - s_n(\mu, f)\|_K^{1/n} < 1, \tag{2.12}$$

where $\| \ \|_K$ denotes the sup norm over K. As an application of Theorem 2.3, we give a characterization of such measures in terms of the quantity ρ in (2.6) under the assumption that (2.9) is satisfied.

THEOREM 2.4. *Assume that the "reflection coefficients" $\Phi_n(0)$ for a measure μ satisfy the condition* (2.9). *Then μ has the geometric convergence property if and only if $\rho < 1$, where ρ is defined in* (2.6).

In particular, the Jacobi measures do not have the geometric convergence property. Theorem 2.4 implies, in fact, that unless the measure is extremely nice (cf. Theorem 2.2), the possible singularities of a function $f \in H^2$ have an adverse effect on the rate of convergence of its orthogonal polynomial expansion inside the unit disc, where f is analytic. This is in keeping with the "contamination principle" described in [12].

3. PROOFS

The proof of Theorem 2.3 and the remarks following it rely upon the following lemma of Blatt, Saff, and Simkani [2]. If $K \subset \mathbf{C}$ is compact, then we denote its logarithmic capacity by $\operatorname{cap}(K)$ and its *equilibrium measure* by ν_K. This equilibrium measure is the unique positive, unit Borel measure supported on K such that

$$\int_K \log|z-t|\, d\nu_K(t) = \log \operatorname{cap}(K), \qquad \text{quasi-everywhere on } K. \tag{3.1}$$

Here, and in the sequel, *quasi-everywhere* will mean everywhere except on a Borel set of capacity zero.

LEMMA 3.1 [2]. *Let K be a compact subset of $\mathbf{C}$ having positive capacity, Λ be an infinite subsequence of positive integers, and*

$$p_n = z^n + \cdots \in \Pi_n, \qquad n \in \Lambda,$$

be a sequence of monic polynomials. Then,

(a) *$\nu(p_n)$ converges to ν_K in the weak* topology if both of the following conditions hold:*

$$\limsup_{\substack{n \to \infty \\ n \in \Lambda}} \|p_n\|_K^{1/n} \leqslant \operatorname{cap}(K) \tag{3.2}$$

and, for each closed set A contained in the union of the bounded (open) components of the complement of the outer boundary of K,

$$\lim_{\substack{n \to \infty \\ n \in \Lambda}} \nu(p_n)(A) = 0. \tag{3.3}$$

(b) *Conversely, if the unbounded component of* $\mathbf{C}\backslash K$ *is regular with respect to the Dirichlet problem,* $\nu(p_n) \xrightarrow{*} \nu_K$ *as* $n \to \infty$, $n \in \Lambda$, *and the zeros of* $\{p_n\}_{n \in \Lambda}$ *are uniformly bounded, then the conditions* (3.2) *and* (3.3) *must hold.*

Proof of Theorem 2.3. We distinguish between two cases. If $\rho = 0$, then Theorem 2.2(b) implies that the sequence $\{\phi_n^*(z)\}$ converges uniformly on compact subsets of $\mathbf{C}$ to an entire function. Therefore, in view of Hurwitz's theorem, for any $\varepsilon > 0$, the number of zeros of ϕ_n^* in the disc $|z| \leqslant 1/\varepsilon$, which is the same as the number of zeros of ϕ_n in the annulus $\varepsilon \leqslant |z| < 1$, is bounded uniformly in n. Theorem 2.3 then follows trivially in this case.

Now let $\rho > 0$. For $n \in \Lambda$ sufficiently large, $\Phi_n(0) \neq 0$ and we write

$$\Psi_n(z) := \Phi_n^*(z)/\overline{\Phi_n(0)}, \qquad n = 1, 2, \ldots. \tag{3.4}$$

Then Ψ_n is a monic polynomial of degree n. We shall prove that $\{\Psi_n\}_{n \in \Lambda}$ satisfies the conditions (3.2) and (3.3) of Lemma 3.1 with $K = C_R$, where

$$R = 1/\rho. \tag{3.5}$$

In view of (2.4) and (2.9), $\limsup_{n \to \infty} \|\Phi_n\|_{C_1}^{1/n} \leqslant 1$. Hence, by Bernstein's lemma (cf. [15, Sect. 4.6]),

$$\limsup_{n \to \infty} \|\Phi_n\|_{C_R}^{1/n} \leqslant R. \tag{3.6}$$

Using (3.6), (2.6), and (2.2b), we conclude that

$$\limsup_{n \to \infty} \|\Phi_n^*\|_{C_R}^{1/n} \leqslant 1. \tag{3.7}$$

In view of (2.7), this implies that

$$\limsup_{\substack{n \to \infty \\ n \in \Lambda}} \|\Psi_n\|_{C_R}^{1/n} \leqslant R. \tag{3.8}$$

Now, R is the capacity of C_R and Ψ_n is a monic polynomial. Thus $\{\Psi_n\}_{n \in \Lambda}$ satisfies the condition (3.2) for $K = C_R$. The condition (3.3) is satisfied in view of Proposition 2.1(b) when $\rho = 1$ and Theorem 2.2(a) in the case when $\rho < 1$. Lemma 3.1 now implies Theorem 2.3. ∎

Next, we demonstrate that if $0 < \Phi_n(0) < 1$, for all $n \geqslant 1$, then (2.8) implies (2.9). Suppose that $0 < \Phi_n(0) < 1$ and (2.9) does not hold. Necessarily, then, $\rho = 1$. Moreover, we observe that (2.2a) implies that $\Phi_n(z)$ and $\Phi_n^*(z)$ are real for real values of z; in particular, $\Phi_n(1) = \Phi_n^*(1)$ for $n = 0, 1, \ldots$. Using (2.2a) again, we see that

$$\Phi_n^*(1) = \Phi_n(1) = \prod_{k=0}^{n-1} (1 + \Phi_k(0)).$$

Since $\log(1+x) \geqslant x/2$ for $0<x<1$, this gives

$$\|\Phi_n\|_{C_1} \geqslant \Phi_n(1) \geqslant \exp\left(\frac{1}{2}\sum_{k=0}^{n-1} \Phi_k(0)\right).$$

Now, the zeros of Φ_n all lie in the open unit disc. Hence, Lemma 3.1(b) implies that if (2.9) does not hold, then (2.8) cannot hold either.

Proof of Theorem 2.4. Let $0<\rho<r<1$. Then (3.7) holds and hence

$$\limsup_{n\to\infty} \|\Phi_n\|_{C_\rho}^{1/n} \leqslant \rho. \tag{3.9}$$

Using Bernstein's lemma [15, Sect. 4.6], we then get

$$\limsup_{n\to\infty} \|\Phi_n\|_{C_r}^{1/n} \leqslant r. \tag{3.10}$$

If $\rho=0$, then (3.7) holds for every R and hence we get (3.10) in this case also. Now, Theorem 2.2(c) implies that $\log \mu'$ is integrable, and hence, the sequence $\{\kappa_n\}$ converges to a positive real number. Thus, (3.10) implies that

$$\limsup_{n\to\infty} \|\phi_n\|_{C_r}^{1/n} \leqslant r. \tag{3.11}$$

It is now elementary to verify that

$$\limsup_{n\to\infty} \|f - s_n(\mu, f)\|_K^{1/n} \leqslant r < 1, \tag{3.12}$$

for every set $K \subseteq \{z : |z| \leqslant r\}$.

Next, let $\rho = 1$. We consider the function

$$f(z) := \sum_{n\in\Lambda} \frac{\overline{\Phi_n(0)}\,\Phi_n(z)}{n^2\,\|\Phi_n\|_{C_1}}. \tag{3.13}$$

Clearly, f is continuous on $|z| \leqslant 1$ and analytic on $|z|<1$; in particular, $f \in H^2(\mu)$. However, if N is any fixed integer in Λ, then

$$|f(0) - s_N(\mu, f, 0)| = \sum_{\substack{n\geqslant N\\ n\in\Lambda}} \frac{|\Phi_n(0)|^2}{n^2\,\|\Phi_n\|_{C_1}} \geqslant \frac{|\Phi_N(0)|^2}{N^2\,\|\Phi_N\|_{C_1}}. \tag{3.14}$$

Thus, since $\rho=1$ and $\lim_{N\to\infty} \|\Phi_N\|_{C_1}^{1/N} = 1$, $N\in\Lambda$ (cf. (3.6) with $R=1$), we have

$$\limsup_{n\to\infty} \|f - s_n(\mu, f)\|_K^{1/n} \geqslant 1 \tag{3.15}$$

for any compact set K such that $0 \in K$. This completes the proof. ∎

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